BINARY CONSTRAINED FLOWS AND SEPARATION OF VARIABLES FOR SOLITON EQUATIONS

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(31 May 2000)

Dedicated to Martin Kruskal's seventy-fifth birthday

Abstract

In contrast to mono-constrained flows with N degrees of freedom, binary constrained flows of soliton equations, admitting 2×2 Lax matrices, have $2N$ degrees of freedom. By means of the existing method, Lax matrices only yield the first N pairs of canonical separated variables. An approach for constructing the second N pairs of canonical separated variables with additional N separated equations is introduced. The Jacobi inversion problems for binary constrained flows are then established. Finally, the separability of binary constrained flows together with the factorization of soliton equations by the spatial and temporal binary constrained flows leads to the Jacobi inversion problems for soliton equations.

1. Introduction

The separation of variables is one of the most universal methods for solving completely integrable models, both classical and quantum. If a finite-dimensional integrable Hamiltonian system (FDIHS) with m degrees of freedom has m functionally independent and involutive integrals of motion P_i, $1 \le i \le m$, the separation of variables [1, 2] means to construct m pairs of canonical variables

$$\{u_k, u_l\} = \{v_k, v_l\} = 0, \ \{v_k, u_l\} = \delta_{kl}, \ 1 \le k, l \le m, \tag{1}$$

and m separated equations

$$f_k(u_k, v_k, P_1, ..., P_m) = 0, \ 1 \le k \le m. \tag{2}$$

Such pairs of variables are called canonical separated variables.

For a FDIHS admitting a 2×2 Lax matrix, there exists a general method to construct canonical separated variables based on the Lax matrix (for example, see [1]-[7]). The corresponding separated equations enable us to express the generating function of canonical transformation in a completely separated form as an Abelian integral on the associated invariant spectral curve. The resulting linearizing map is essentially the Abel map to the Jacobi variety of the spectral curve, thereby providing a link with the algebro-geometric linearization method [8]. An important feature of the separation of variables for a FDIHS is that the number of pairs of canonical separated variables must be equal to the number of degrees of freedom. However, in some cases, it is found that the existing method may not yield enough pairs of canonical separated variables. It has been a challenging problem [1] how to construct additional canonical separated variables which are required for separation of variables.

Binary constrained flows of soliton hierarchies, recently attracting much attention (for example, see [10]-[13]), are such specific cases, which need to be handled by a different approach. The degree of freedom of binary constrained flows admitting 2×2 Lax matrices is $2N$. By using the existing method [1, 2], the Lax matrices allow to directly construct the first N pairs of canonical separated variables $u_1, ..., u_N$ and $v_1, ..., v_N$. In this report, we would like to show an approach for determining the second N pairs of canonical separated variables and N additional separated equations for binary constrained flows. The crucial point is to construct a new set of generating functions $\widetilde{B}(\lambda)$ and $\widetilde{A}(\lambda)$ defining $u_{N+1}, ..., u_{2N}$ by the set of zeros of $\widetilde{B}(\lambda)$ and $v_{N+k} = \widetilde{A}(u_{N+k})$, $1 \leq k \leq N$. To keep the canonical conditions (1) and obtain the separated equations (2), it is found that certain commutator relations need to be imposed on $\widetilde{B}(\lambda)$ and $\widetilde{A}(\lambda)$, and $\widetilde{A}(\lambda)$ has some link with the common generating function of integrals of motion of binary constrained flows, which also provides a clue to construct the $\widetilde{B}(\lambda)$ and $\widetilde{A}(\lambda)$. Having analyzed the separation of variables, the Jacobi inversion problems can be naturally presented for binary constrained flows.

The separation of variables for soliton equations consists of two steps of separation of variables [7]. The first step is to factorize $1 + 1$ dimensional soliton equations into two commuting spatial and temporal FDIHSs resulted from the spatial and temporal binary constrained flows. The second step is to analyze the separation of variables for the spatial and temporal binary constrained flows to produce their Jacobi inversion problems. Finally, combining the factorization of soliton equations with the Jacobi inversion problems for the spatial and temporal binary constrained flows enables us to establish the Jacobi inversion problems for soliton equations. We will use the AKNS equations [9] to illustrate the whole process. Of course, the approach adopted can be applied to the whole AKNS hierarchy and other soliton hierarchies.

2. Separation of variables for binary constrained flows

Let us first describe binary constrained flows admitting 2×2 Lax matrices, and then show an approach for constructing $2N$ pairs of canonical separated variables.

Assume that a soliton hierarchy

$$u_{t_n} = K_n(u) = J \frac{\delta \tilde{H}_n}{\delta u}, \ u = (u_1, ..., u_q)^T, \ n \geq 0, \tag{3}$$

where J is a Hamiltonian operator, is determined by a spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi, \ U = (U_{ij})_{2 \times 2}, \ \phi = (\phi_1, \phi_2)^T, \tag{4}$$

and the associated spectral problems

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, u_x, ...; \lambda)\phi, \ V^{(n)} = (V_{ij}^{(n)})_{2 \times 2}.$$

The compatability conditions of the adjoint spectral problem

$$\psi_x = -U^T(u, \lambda)\psi, \ \psi = (\psi_1, \psi_2)^T \tag{5}$$

and the adjoint associated spectral problems

$$\psi_{t_n} = -V^{(n)T}\psi = -V^{(n)T}(u, u_x, ...; \lambda)\psi$$

still give rise to the same soliton hierarchy (3).

Upon introducing N distinct eigenvalues $\lambda_1, ..., \lambda_N$, we have the spatial system

$$\phi_x^{(j)} = U(u, \lambda_j)\phi^{(j)}, \ \psi_x^{(j)} = -U^T(u, \lambda_j)\psi^{(j)}, \tag{6}$$

where $\phi^{(j)} = (\phi_{1j}, \phi_{2j})^T, \ \psi^{(j)} = (\psi_{1j}, \psi_{2j})^T, 1 \leq j \leq N$, and the temporal system

$$\phi_t^{(j)} = V^{(n)}(u, \lambda_j)\phi^{(j)}, \ \psi_t^{(j)} = -V^{(n)T}(u, \lambda_j)\psi^{(j)}, \tag{7}$$

where $1 \leq j \leq N$. Let us take the Bargmann symmetry constraint

$$K_0 = J \sum_{j=1}^{N} E_j J \frac{\delta \lambda_j}{\delta u} = J \sum_{j=1}^{N} \psi^{(j)T} \frac{\partial U(u, \lambda_j)}{\partial u} \phi^{(j)}, \tag{8}$$

where the E_j are normalized constants, and suppose that (8) has an inverse function

$$u = f(\xi_1, ..., \xi_q), \ \xi_i = \sum_{j=1}^{N} \psi^{(j)T} \frac{\partial U(u, \lambda_j)}{\partial u_i} \phi^{(j)}, \ 1 \leq i \leq q. \tag{9}$$

Replacing u with f in N replicas of (6) and (7), we obtained the so-called spatial constrained flow

$$\phi_x^{(j)} = U(f, \lambda_j)\phi^{(j)}, \ \psi_x^{(j)} = -U^T(f, \lambda_j)\psi^{(j)}, \tag{10}$$

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and the so-called temporal constrained flow

$$\phi_t^{(j)} = V^{(n)}(f, f_x, ...; \lambda_j)\phi^{(j)}, \ \psi_t^{(j)} = -V^{(n)T}(f, f_x, ...; \lambda_j)\psi^{(j)}, \qquad (11)$$

where $1 \leq j \leq N$. Now if ϕ_{ij} and ψ_{ij} solve two constrained flows, then $u = f(\xi_1, ..., \xi_q)$ gives rise to a solution to the soliton equation $u_{t_n} = K_n(u)$. The above manipulation is called binary nonlinearization [10, 14].

It is known that constrained flows (CFs) have natural Lax matrices generated from a solution

$$M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

to $M_x = [U, M]$ and $M_{t_n} = [V^{(n)}, M]$ (for example, see [15, 16]). To determine $2N$ pairs of canonical separated variables for binary CFs, based on Lax matrices $M(\lambda)$, we search for two sets of generating functions $\overline{A}(\lambda), \overline{B}(\lambda)$ and $\widetilde{A}(\lambda), \widetilde{B}(\lambda)$ such that

$$\begin{cases} \{\overline{B}(\lambda), \overline{B}(\mu)\} = \{\widetilde{B}(\lambda), \widetilde{B}(\mu)\} = \{\overline{A}(\lambda), \overline{A}(\mu)\} = \{\widetilde{A}(\lambda), \widetilde{A}(\mu)\} = 0, \\ \{\overline{B}(\lambda), \widetilde{B}(\mu)\} = \{\overline{B}(\lambda), \widetilde{A}(\mu)\} = \{\widetilde{B}(\lambda), \overline{A}(\mu)\} = \{\overline{A}(\lambda), \widetilde{A}(\mu)\} = 0, \\ \{\overline{A}(\lambda), \overline{B}(\mu)\} = \dfrac{\overline{B}(\mu) - \overline{B}(\lambda)}{\lambda - \mu}, \ \{\widetilde{A}(\lambda), \widetilde{B}(\mu)\} = \dfrac{\widetilde{B}(\mu) - \widetilde{B}(\lambda)}{\lambda - \mu}, \end{cases}$$
$$(12)$$

under the standard Poisson bracket

$$\{F, G\} = \sum_{i=1}^{2} \sum_{j=1}^{N} \left(\frac{\partial F}{\partial \psi_{ij}} \frac{\partial G}{\partial \phi_{ij}} - \frac{\partial F}{\partial \phi_{ij}} \frac{\partial G}{\partial \psi_{ij}} \right). \qquad (13)$$

Such two sets of generating functions can be constructed from Lax matrices $M(\lambda)$ and a common generating function of integrals of motion for binary CFs. We expect each pair of generating functions can yield N pairs of canonical separated variables, through defining $u_1, ..., u_N$ by the set of zeros of $\overline{B}(\lambda)$, $u_{N+1}, ..., u_{2N}$ by the set of zeros of $\widetilde{B}(\lambda)$, and

$$v_k = \overline{A}(u_k), \ v_{N+k} = \widetilde{A}(u_{N+k}), \ 1 \leq k \leq N, \qquad (14)$$

which will also give us all $2N$ separated equations. Therefore, the separation of variables for binary CFs becomes the problem to find two sets of generating functions satisfying the above commutator relations (12). The whole process will be illustrated by the AKNS equations.

3. Binary constrained flows of the AKNS equations

Let us start from the AKNS spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi, \ U = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \ u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (15)$$

and take the associated spectral problems

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, \lambda)\phi, \ V^{(n)} = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i}, \qquad (16)$$

with a_i, b_i, c_i being defined by

$$a_0 = -1, \ b_0 = c_0 = 0, \ a_1 = 0, \ b_1 = q, \ c_1 = r, \ a_2 = \tfrac{1}{2}qr, \ ...,$$

$$\begin{pmatrix} c_{k+1} \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} c_k \\ b_k \end{pmatrix}, \ a_{k+1} = \partial^{-1}(qc_{k+1} - rb_{k+1}), \ k \geq 1,$$

where L is given by

$$L = \frac{1}{2} \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \end{pmatrix}.$$

The compatibility conditions of (15) and (16) give the AKNS hierarchy

$$u_{t_n} = J \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = J\frac{\delta \tilde{H}_{n+1}}{\delta u}, \ J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \ \tilde{H}_n = \int \frac{2a_{n+1}}{n+1}dx, \ n \geq 1,$$

$$\qquad (17)$$

which contains the AKNS equations

$$q_{t_2} = -\frac{1}{2}q_{xx} + q^2 r, \ r_{t_2} = \frac{1}{2}r_{xx} - r^2 q. \qquad (18)$$

Introducing N distinct eigenvalues $\lambda_j, \ 1 \leq j \leq N$, we have

$$\begin{cases} \Phi_{1x} = -\Lambda\Phi_1 + q\Phi_2, \ \Phi_{2x} = r\Phi_1 + \Lambda\Phi_2, \\ \Psi_{1x} = \Lambda\Psi_1 - r\Psi_2, \ \Psi_{2x} = -q\Psi_1 - \Lambda\Psi_2, \end{cases} \qquad (19)$$

and the Bargmann symmetry constraint reads as

$$\frac{\delta H_1}{\delta u} - \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} r \\ q \end{pmatrix} - \begin{pmatrix} \langle \Psi_1, \Phi_2 \rangle \\ \langle \Psi_2, \Phi_1 \rangle \end{pmatrix} = 0, \qquad (20)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^N$ and

$$\Phi_i = (\phi_{i1}, ..., \phi_{iN})^T, \ \Psi_i = (\psi_{i1}, ..., \psi_{iN})^T, \ i = 1, 2, \ \Lambda = \text{diag}(\lambda_1, ..., \lambda_N).$$

Therefore, the spatial constrained flow (10) is the following x-FDIHS [10]

$$\Phi_{1x} = \frac{\partial F_1}{\partial \Psi_1}, \ \Phi_{2x} = \frac{\partial F_1}{\partial \Psi_2}, \ \Psi_{1x} = -\frac{\partial F_1}{\partial \Phi_1}, \ \Psi_{2x} = -\frac{\partial F_1}{\partial \Phi_2}, \qquad (21)$$

with the Hamiltonian

$$F_1 = \langle \Lambda\Psi_2, \Phi_2 \rangle - \langle \Lambda\Psi_1, \Phi_1 \rangle + \langle \Psi_2, \Phi_1 \rangle \langle \Psi_1, \Phi_2 \rangle.$$

Under the symmetry constraint (20) and the x-FDIHS (21), the binary t_2-constrained flow (11) can be transformed into the following t_2-FDIHS

$$\Phi_{1t_2} = \frac{\partial F_2}{\partial \Psi_1}, \ \Phi_{2t_2} = \frac{\partial F_2}{\partial \Psi_2}, \ \Psi_{1t_2} = -\frac{\partial F_2}{\partial \Phi_1}, \ \Psi_{2t_2} = -\frac{\partial F_2}{\partial \Phi_2}, \qquad (22)$$

with the Hamiltonian

$$F_2 = \langle \Lambda^2 \Psi_2, \Phi_2 \rangle - \langle \Lambda^2 \Psi_1, \Phi_1 \rangle + \langle \Psi_2, \Phi_1 \rangle \langle \Lambda \Psi_1, \Phi_2 \rangle +$$

$$\langle \Lambda \Psi_2, \Phi_1 \rangle \langle \Psi_1, \Phi_2 \rangle - \tfrac{1}{2} (\langle \Psi_2, \Phi_2 \rangle - \langle \Psi_1, \Phi_1 \rangle) \langle \Psi_2, \Phi_1 \rangle \langle \Psi_1, \Phi_2 \rangle.$$

The Lax matrix $M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$ for the FDIHSs (21) and (22) is given by [17]

$$A(\lambda) = -1 + \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j}}{2(\lambda - \lambda_j)}, \ B(\lambda) = \sum_{j=1}^{N} \frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j}, \ C(\lambda) = \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{2j}}{\lambda - \lambda_j}. \qquad (23)$$

A straightforward calculation yields

$$P(\lambda) := A^2(\lambda) + B(\lambda)C(\lambda) = 1 + \sum_{j=1}^{N} [\frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2}], \qquad (24)$$

where the P_j and P_{N+j} are $2N$ involutive integrals of motion for (21) and (22)

$$P_j = \frac{1}{2} \sum_{k \neq j} \frac{1}{\lambda_j - \lambda_k} [(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k})$$

$$+ \psi_{2j}\phi_{2j} - \psi_{1j}\phi_{1j} + 4\psi_{1j}\phi_{2j}\psi_{2k}\phi_{1k}], \ 1 \leq j \leq N, \qquad (25)$$

$$P_{N+j} = \frac{1}{2}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}), \ 1 \leq j \leq N. \qquad (26)$$

It is easy to verify that

$$F_1 = \sum_{j=1}^{N} (\lambda_j P_j + P_{N+j}^2) - (\sum_{j=1}^{N} \frac{P_j}{2})^2, \qquad (27)$$

$$F_2 = \sum_{j=1}^{N} (\lambda_j^2 P_j + 2\lambda_j P_{N+j}^2) - (\sum_{j=1}^{N} \frac{P_j}{2}) \sum_{j=1}^{N} (\lambda_j P_j + P_{N+j}^2) + (\sum_{j=1}^{N} \frac{P_j}{2})^3. \qquad (28)$$

With respect to the standard Poisson bracket (13), it is found [17] that

$$\begin{cases} \{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0, \\ \{A(\lambda), B(\mu)\} = \frac{1}{\lambda-\mu}[B(\mu) - B(\lambda)], \\ \{A(\lambda), C(\mu)\} = \frac{1}{\lambda-\mu}[C(\lambda) - C(\mu)], \\ \{B(\lambda), C(\mu)\} = \frac{2}{\lambda-\mu}[A(\mu) - A(\lambda)]. \end{cases} \qquad (29)$$

Then $\{A^2(\lambda) + B(\lambda)C(\lambda), A^2(\mu) + B(\mu)C(\mu)\} = 0$ implies that the integrals of motion P_j and P_{N+j}, $1 \le j \le N$, are in involution in pairs. The AKNS equations (18) are factorized by the x-FDIHS (21) and the t_2-FDIHS (22). Namely, if Φ_1, Φ_2, Ψ_1 and Ψ_2 solve the x-FDIHS (21) and the t_2-FDIHS (22) simultaneously, then (q, r) given by (20) solves the AKNS equations (18).

4. Separation of variables for the AKNS equations

The commutator relations (29) and a common generating function of integrals of motion

$$\frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j} \psi_{1j} + \phi_{2j} \psi_{2j}}{\lambda - \lambda_j} = \sum_{j=1}^{N} \frac{P_{N+j}}{\lambda - \lambda_j}$$

enable us to construct two sets of generating functions which lead to $2N$ pairs of canonical separated variables. The required two sets of generating functions for the AKNS equations (18) are the following

$$\overline{A}(\lambda) = B(\lambda) - A(\lambda) - \widetilde{A}(\lambda) = 1 + \sum_{j=1}^{N} \frac{(\psi_{2j} - \psi_{1j})\phi_{1j}}{\lambda - \lambda_j}, \tag{30}$$

$$\overline{B}(\lambda) = B(\lambda) - 2A(\lambda) - C(\lambda) = 2 + \sum_{j=1}^{N} \frac{(\psi_{2j} - \psi_{1j})(\phi_{1j} + \phi_{2j})}{\lambda - \lambda_j}, \tag{31}$$

$$\widetilde{A}(\lambda) = \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}}{\lambda - \lambda_j}, \quad \widetilde{B}(\lambda) = 1 + \frac{1}{2} \sum_{j=1}^{N} \frac{(\phi_{1j} + \phi_{2j})^2}{\lambda - \lambda_j}. \tag{32}$$

Let us now introduce u_k, u_{N+k}, $1 \le k \le N$, by

$$\overline{B}(\lambda) = 2 \frac{\overline{R}(\lambda)}{K(\lambda)}, \quad \widetilde{B}(\lambda) = \frac{\widetilde{R}(\lambda)}{K(\lambda)}, \tag{33}$$

where $\overline{R}(\lambda)$, $\widetilde{R}(\lambda)$ and $K(\lambda)$ read as

$$\overline{R}(\lambda) = \prod_{k=1}^{N} (\lambda - u_k), \quad \widetilde{R}(\lambda) = \prod_{k=1}^{N} (\lambda - u_{N+k}), \quad K(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k). \tag{34}$$

A direct computation can show the following result.

THEOREM 4.1. *Assume that* $\lambda_j, \phi_{ij}, \psi_{ij}$, $i = 1, 2$, $1 \le j \le N$, *are all real, and* $u_1, ..., u_N$ *are single zeros of* $\overline{B}(\lambda)$. *Then the variables* $u_1, ..., u_{2N}$ *defined by (33) and (34), and the variables* $v_1, ..., v_{2N}$ *defined by the corresponding formula (14) are canonically conjugated, i.e., they satisfy the commutator relations (1) with* $m = 2N$.

It follows from (33) and (34) that

$$(\psi_{2j} - \psi_{1j})(\phi_{1j} + \phi_{2j}) = 2\frac{\overline{R}(\lambda_j)}{K'(\lambda_j)}, \ (\phi_{1j} + \phi_{2j})^2 = 2\frac{\widetilde{R}(\lambda_j)}{K'(\lambda_j)}, \ 1 \le j \le N,$$

which leads to

$$(\phi_{1j} + \phi_{2j}) = \sqrt{\frac{2\widetilde{R}(\lambda_j)}{K'(\lambda_j)}}, \ (\psi_{2j} - \psi_{1j}) = \frac{\sqrt{2}\,\overline{R}(\lambda_j)}{\sqrt{\widetilde{R}(\lambda_j)K'(\lambda_j)}}, \ 1 \le j \le N. \quad (35)$$

By substituting u_k into $\overline{A}(\lambda)$ of (30), u_{N+k} into $\widetilde{A}(\lambda)$ of (32) and noting

$$(A(\lambda) - B(\lambda))^2 - \overline{B}(\lambda)B(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda) = P(\lambda),$$

one gets the separated equations

$$v_k = B(u_k) - A(u_k) - \widetilde{A}(u_k) = \sqrt{P(u_k)} - \widetilde{A}(u_k)$$

$$= \sqrt{1 + \sum_{j=1}^{N}[\frac{P_j}{u_k - \lambda_j} + \frac{P_{N+j}^2}{(u_k - \lambda_j)^2}]} - \sum_{j=1}^{N}\frac{P_{N+j}}{u_k - \lambda_j}, \ 1 \le k \le N, \quad (36)$$

$$v_{N+k} = \widetilde{A}(u_{N+k}) = \sum_{j=1}^{N}\frac{P_{N+j}}{u_{N+k} - \lambda_j}, \ 1 \le k \le N. \quad (37)$$

Replacing v_k by the partial derivative $\frac{\partial S}{\partial u_k}$ of the generating function S of canonical transformation and interpreting the P_j and P_{N+j} as integration constants, the above separated equations give rise to the Hamilton-Jacobi equations which are completely separated and can be integrated to give the completely separated solution for S

$$S(u_1, ..., u_{2N}) = \sum_{k=1}^{N}[\int^{u_k}(\sqrt{P(\lambda)} - \widetilde{A}(\lambda))d\lambda + \int^{u_{N+k}}\widetilde{A}(\lambda)d\lambda]$$

$$= \sum_{k=1}^{N}[\int^{u_k}\sqrt{P(\lambda)}d\lambda - \sum_{j=1}^{N}P_{N+j}\ln|\frac{u_k - \lambda_j}{u_{N+k} - \lambda_j}|]. \quad (38)$$

The linearizing coordinates are then

$$\begin{cases} Q_j = \dfrac{\partial S}{\partial P_j} = \dfrac{1}{2}\sum_{k=1}^{N}\int^{u_k}\dfrac{1}{(\lambda - \lambda_j)\sqrt{P(\lambda)}}d\lambda, 1 \le j \le N, \\[4mm] Q_{N+j} = \dfrac{\partial S}{\partial P_{N+j}} = \sum_{k=1}^{N}[\int^{u_k}\dfrac{P_{N+j}}{(\lambda - \lambda_j)^2\sqrt{P(\lambda)}}d\lambda - \ln|\dfrac{u_k - \lambda_j}{u_{N+k} - \lambda_j}|], \end{cases}$$
$$(39)$$

where $1 \le j \le N$. These coordinates Q_j and Q_{N+j}, $1 \le j \le N$, constitute the action-angle variables together with the P_j and P_{N+j}, $1 \le j \le N$. By

using (27) and (28), the linear flows induced by the x-FDIHS (21) and the t_2-FDIHS (22) lead to the Jacobi inversion problem for the x-FDIHS (21)

$$\begin{cases} 2Q_j = \gamma_j + (2\lambda_j - \sum_{k=1}^{N} P_k)x, \\ Q_{N+j} = \gamma_{N+j} + 2P_{N+j}x, \end{cases} \tag{40}$$

and the Jacobi inversion problem for the t_2-FDIHS (22)

$$\begin{cases} 2Q_j = \bar{\gamma}_j + [2\lambda_j^2 - \sum_{k=1}^{N}(\lambda_k P_k + \lambda_j P_k + P_{N+k}^2) + \frac{3}{4}(\sum_{k=1}^{N} P_k)^2]t_2, \\ Q_{N+j} = \bar{\gamma}_{N+j} + P_{N+j}(4\lambda_j - \sum_{k=1}^{N} P_k)t_2, \end{cases} \tag{41}$$

where $1 \leq j \leq N$, the Q_j and Q_{N+j} are defined by (39), and γ_j and $\bar{\gamma}_j$, $1 \leq j \leq 2N$, are arbitrary constants.

Since the AKNS equations (18) are factorized by the x-FDIHS (21) and the t_2-FDIHS (22), combining the Jacobi inversion problems (40) and (41) together gives rise to the following theorem.

THEOREM 4.2. *The AKNS equations (18) have the Jacobi inversion problem determined by*

$$\sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_j)\sqrt{P(\lambda)}} d\lambda = \tilde{\gamma}_j + (2\lambda_j - \sum_{k=1}^{N} P_k)x$$

$$+ [2\lambda_j^2 - \sum_{k=1}^{N}(\lambda_k P_k + \lambda_j P_k + P_{N+k}^2) + \frac{3}{4}(\sum_{k=1}^{N} P_k)^2]t_2,$$

$$\sum_{k=1}^{N} [\int^{u_k} \frac{P_{N+j}}{(\lambda - \lambda_j)^2\sqrt{P(\lambda)}} d\lambda - ln \mid \frac{u_k - \lambda_j}{u_{N+k} - \lambda_j} \mid]$$

$$= \tilde{\gamma}_{N+j} + 2P_{N+j}x + P_{N+j}(4\lambda_j - \sum_{k=1}^{N} P_k)t_2,$$

where $1 \leq j \leq N$, and $\tilde{\gamma}_j$ and $\tilde{\gamma}_{N+j}$, $1 \leq j \leq N$, are arbitrary constants.

We remark that the above Jacobi inversion problem for the AKNS equations (18) is different from that in [18], which was generated from another class of canonical separated variables for the binary constrained flows (21) and (22). The above manipulation may also be similarly made for the whole AKNS hierarchy, and the approach depicted in Section 2 can be applied to other soliton hierarchies such as the KdV hierarchy and the Kaup-Newell hierarchy.

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